On the size of graphs without repeated cycle lengths ⋆

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Abstract

In 1975, P. Erdös proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $t = 1260r + 169 \;\; (r \geq 1)$ and $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$. Consequently, $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{7654}{19071}}$, which is better than the previous bounds $\sqrt{2}$ [Y. Shi, Discrete Math. 71(1988), 57-71], $\sqrt{2.4}$ [C. Lai, Australas. J. Combin. 27(2003), 101-105]. The conjecture $\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}$ is not true.

Key words: Graph, cycle, number of edges.
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1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdös raised the problem of

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determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi [15] proved a lower bound.

Theorem 1 (Shi [15])

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$.

Chen, Lehel, Jacobson, and Shreve [3], Jia [5], Lai [6–8], Shi [16,18–20] obtained some additional related results.

Boros, Caro, Füredi and Yuster [2] proved an upper bound.

Theorem 2 (Boros, Caro, Füredi and Yuster [2]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [9] improved the lower bound.

Theorem 3 (Lai [9])

$$f(n) \geq n + \sqrt{2.4}\sqrt{n}(1-o(1))$$

Lai [6,9] proposed the following conjectures:

Conjecture 4 (Lai [9])

$$\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}.$$

Conjecture 5 (Lai [6])

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

Markström [13] raised the following problem:

Problem 6 (Markström [13]) Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Results for the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length can be found in [2,3,15].

Survey articles on this problem can be found in Tian [21], Zhang [24], Lai and Liu [10].

The progress of all 50 problems in [1] can be find in Locke [12].

A related topic concerns the Entringer problem, which is to determine which simple graphs have exactly one cycle of each length l, $3 \leq l \leq v$ (see problem 10 in [1]). This problem was posed in 1973 by R. C. Entringer. For developments on this topic, see [4,11,13,14,17,22,23].

In this paper, we construct a graph G having no two cycles with the same length which leads to the following result.

Theorem 7 Let $t = 1260r + 169\ \ (r \geq 1)$, then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$.

Hence conjecture 4 is not true.

2 Proof of theorem 7

Proof. Let $t = 1260r + 169, r \geq 1$, $n_t = \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$, $n \geq n_t$. We shall show that there exists a graph G on n vertices with $n + \frac{107}{3}t + \frac{7}{3}$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq 20t, 27t \leq i \leq 28t + 64, 29t - 734 \leq i \leq 29t + 267, 30t - 531 \leq i \leq 30t + 57, 31t - 741 \leq i \leq 31t + 58, 32t - 740 \leq i \leq 32t + 57, 33t - 741 \leq i \leq 33t + 57, 34t - 741 \leq i \leq 34t + 52, 35t - 746 \leq i \leq 35t + 60, 36t - 738 \leq i \leq 36t + 60, 37t - 738 \leq i \leq 37t + 799, i = 20t + j(1 \leq j \leq \frac{t-7}{6}), i = 20t + \frac{t-1}{6} + j(1 \leq j \leq \frac{t-7}{6}), i = 21t + 2j + 1(0 \leq j \leq t-1), i = 21t + 2j(0 \leq j \leq \frac{t-1}{2}), i = 23t + 2j + 1(0 \leq j \leq \frac{t-1}{2})$, and $i = 20t + \frac{t-1}{6}$, $i = 20t + \frac{t-1}{3} + \frac{t-1}{6} - 1$, $i = 20t + \frac{t-1}{3} + \frac{t-1}{6}$, $i = 20t + \frac{2t-2}{3}$, $i = 21t - 2$, $i = 21t - 1$, $i = 26t)$.

Now we define these B_is. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq \frac{t-7}{6}$, let the subgraph B_{20t+i} consists of a cycle

$$xa_i^1 a_i^2 ... a_i^{\frac{62t-8}{3}+2i} x$$

and a path:

$$xa_{i,1}^{1}a_{i,1}^{2}...a_{i,1}^{\frac{59t-5}{6}}a_{i}^{\frac{61t-1}{6}+i}$$

Based the construction, B_{20t+i} contains exactly three cycles of lengths:

$$20t+i, 20t+\frac{t-1}{3}+i-1, 20t+\frac{2t-2}{3}+2i-1.$$

For $1 \leq i \leq \frac{t-7}{6}$, let the subgraph $B_{20t+\frac{t-1}{6}+i}$ consists of a cycle

$$xb_{i}^{1}b_{i}^{2}...b_{i}^{\frac{62t-5}{3}+2i}x$$

and a path:

$$xb_{i,1}^{1}b_{i,1}^{2}...b_{i,1}^{10t-1}b_{i}^{\frac{61t-1}{6}+i}$$

Based the construction, $B_{20t+\frac{t-1}{6}+i}$ contains exactly three cycles of lengths:

$$20t+\frac{t-1}{6}+i, 20t+\frac{t-1}{3}+\frac{t-1}{6}+i, 20t+\frac{2t-2}{3}+2i.$$

For $0 \leq i \leq t-1$, let the subgraph $B_{21t+2i+1}$ consists of a cycle

$$xu_{i}^{1}u_{i}^{2}...u_{i}^{25t+2i-1}x$$

and a path:

$$xu_{i,1}^{1}u_{i,1}^{2}...u_{i,1}^{(19t+2i-1)/2}u_{i}^{(23t+2i+1)/2}$$

Based the construction, $B_{21t+2i+1}$ contains exactly three cycles of lengths:

$$21t+2i+1, 23t+2i, 25t+2i.$$

For $0 \leq i \leq \frac{t-3}{2}$, let the subgraph B_{21t+2i} consists of a cycle

$$xv_{i}^{1}v_{i}^{2}...v_{i}^{25t+2i}x$$

and a path:

$$xv_{i,1}^{1}v_{i,1}^{2}...v_{i,1}^{9t+i-1}v_{i}^{12t+i}$$

Based the construction, B_{21t+2i} contains exactly three cycles of lengths:

$$21t+2i, 22t+2i+1, 25t+2i+1.$$

For $i=\frac{t-1}{2}$, B_{21t+2i} is simply a cycle of length $22t-1$.

For $0 \leq i \leq \frac{t-3}{2}$, let the subgraph $B_{23t+2i+1}$ consists of a cycle

$$xw_i^1 w_i^2 ... w_i^{26t+2i+1} x$$

and a path:

$$xw_{i,1}^1 w_{i,1}^2 ... w_{i,1}^{(21t+2i-1)/2} w_i^{(25t+2i+1)/2}$$

Based the construction, $B_{23t+2i+1}$ contains exactly three cycles of lengths:

$$23t+2i+1, 24t+2i+2, 26t+2i+2.$$

For $i=\frac{t-1}{2}$, $B_{23t+2i+1}$ is simply a cycle of length $24t$.

For $58 \leq i \leq t-742$, let the subgraph $B_{27t+i-57}$ consists of a cycle

$$C_{27t+i-57} = xy_i^1 y_i^2 ... y_i^{132t+11i+893} x$$

and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{27t+i-57}$:

$$\begin{gathered}
xy_{i,1}^1 y_{i,1}^2 ... y_{i,1}^{(17t-1)/2} y_i^{(37t-115)/2+i} \\
xy_{i,2}^1 y_{i,2}^2 ... y_{i,2}^{(19t-1)/2} y_i^{(57t-103)/2+2i} \\
xy_{i,3}^1 y_{i,3}^2 ... y_{i,3}^{(19t-1)/2} y_i^{(77t+315)/2+3i} \\
xy_{i,4}^1 y_{i,4}^2 ... y_{i,4}^{(21t-1)/2} y_i^{(97t+313)/2+4i} \\
xy_{i,5}^1 y_{i,5}^2 ... y_{i,5}^{(21t-1)/2} y_i^{(117t+313)/2+5i} \\
xy_{i,6}^1 y_{i,6}^2 ... y_{i,6}^{(23t-1)/2} y_i^{(137t+311)/2+6i} \\
xy_{i,7}^1 y_{i,7}^2 ... y_{i,7}^{(23t-1)/2} y_i^{(157t+309)/2+7i} \\
xy_{i,8}^1 y_{i,8}^2 ... y_{i,8}^{(25t-1)/2} y_i^{(177t+297)/2+8i} \\
xy_{i,9}^1 y_{i,9}^2 ... y_{i,9}^{(25t-1)/2} y_i^{(197t+301)/2+9i} \\
xy_{i,10}^1 y_{i,10}^2 ... y_{i,10}^{(27t-1)/2} y_i^{(217t+305)/2+10i}.
\end{gathered}$$

As a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{27t+i-57}$ contains exactly 66 cycles of lengths:

$27t+i-57$, $28t+i+7$, $29t+i+210$, $30t+i$,
$31t+i+1$, $32t+i$, $33t+i$, $34t+i-5$,
$35t+i+3$, $36t+i+3$, $37t+i+742$, $38t+2i-51$,
$38t+2i+216$, $40t+2i+209$, $40t+2i$, $42t+2i$,
$42t+2i-1$, $44t+2i-6$, $44t+2i-3$, $46t+2i+5$,
$46t+2i+744$, $48t+3i+158$, $49t+3i+215$, $50t+3i+209$,
$51t+3i-1$, $52t+3i-1$, $53t+3i-7$, $54t+3i-4$,
$55t+3i-1$, $56t+3i+746$, $59t+4i+157$, $59t+4i+215$,
$61t+4i+208$, $61t+4i-2$, $63t+4i-7$, $63t+4i-5$,
$65t+4i-2$, $65t+4i+740$, $69t+5i+157$, $70t+5i+214$,
$71t+5i+207$, $72t+5i-8$, $73t+5i-5$, $74t+5i-3$,
$75t+5i+739$, $80t+6i+156$, $80t+6i+213$, $82t+6i+201$,
$82t+6i-6$, $84t+6i-3$, $84t+6i+738$, $90t+7i+155$,
$91t+7i+207$, $92t+7i+203$, $93t+7i-4$, $94t+7i+738$,
$101t+8i+149$, $101t+8i+209$, $103t+8i+205$, $103t+8i+737$,
$111t+9i+151$, $112t+9i+211$, $113t+9i+946$, $122t+10i+153$,
$122t+10i+952$, $132t+11i+894$.

B_0 is a path with an end vertex x and length $n-n_t$. Other B_i is simply a cycle of length i.

Then $f(n) \geq n+\frac{107}{3}t+\frac{7}{3}$, for $n \geq n_t$. This completes the proof.

From Theorem 7, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}},$$

which is better than the previous bounds $\sqrt{2}$ (see [15]), $\sqrt{2+\frac{2}{5}}$ (see [9]).

So Conjecture 4 is not true.

Combining this with Boros, Caro, Füredi and Yuster's upper bound, namely

Theorem 2, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{7654}{19071}}.$$

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